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British Mathematical Olympiad Round 1 2023

Teachers are encouraged to distribute copies of this report to candidates.

Markers' report

The 2023 paper

Olympiad marking

Both candidates and their teachers will find it helpful to know something of the general principles involved in marking Olympiad-type papers. These preliminary paragraphs therefore serve as an exposition of the 'philosophy' which has guided both the setting and marking of all such papers at all age levels, both nationally and internationally.

What we are looking for are full solutions to problems. This involves identifying a suitable strategy, explaining why your strategy solves the problem, and then carrying it out to produce an answer or prove the required result. In marking each question, we look at the solution synoptically and decide whether the candidate has a viable overall strategy or not. An answer which is essentially a solution will be awarded near maximum credit, with marks deducted for errors of calculation, flaws in logic, omission of cases or technical faults. On the other hand, an answer which does not present a complete argument is marked on a '0 plus' basis; up to 4 marks might be awarded for particular cases or insights. In general the logical structure of the mark scheme aims to reflect the logical structure of the problem while rewarding correct arguments more generously than correct calculations.

This approach is therefore rather different from what happens in public examinations such as GCSE, AS and A level, where credit is given for the ability to carry out individual techniques regardless of how these techniques fit into a protracted argument. It is therefore vital that candidates taking Olympiad papers realise the importance of the comment in the rubric about trying to finish whole questions rather than attempting lots of disconnected parts.

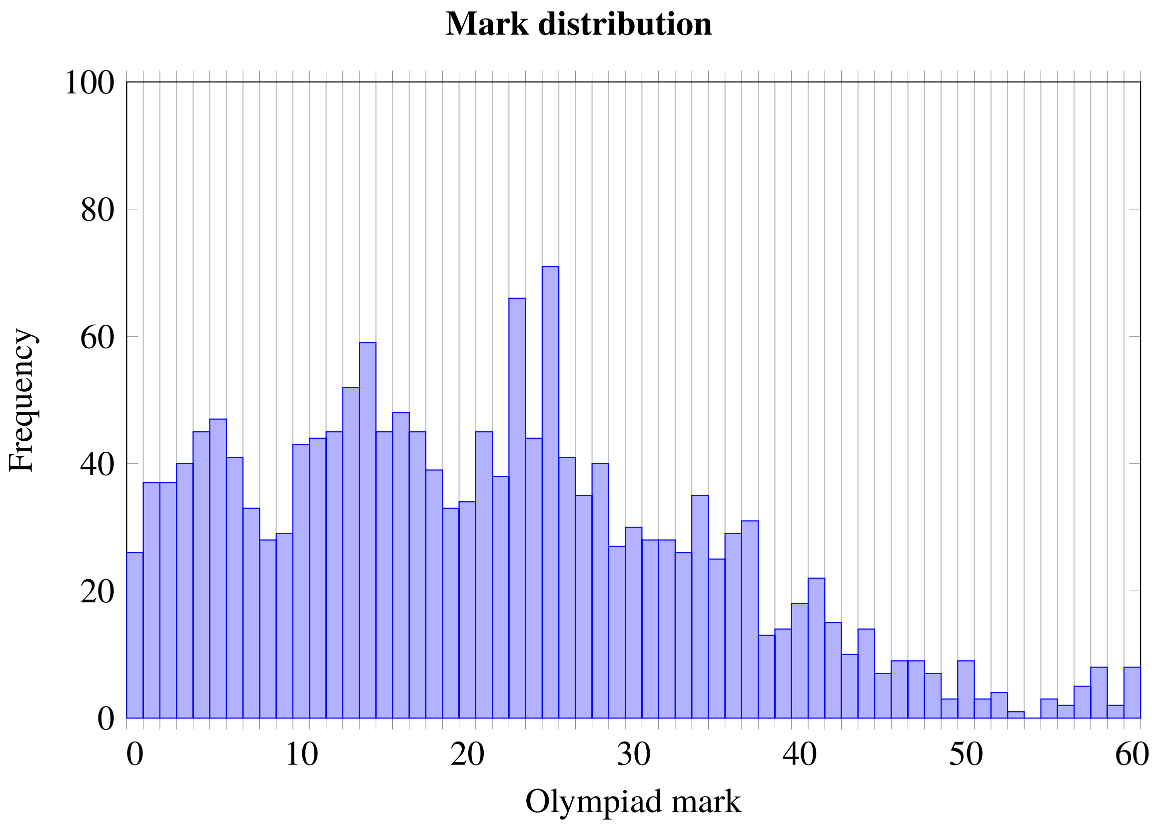
General comments

Candidates found this to be one of the most accessible BMO1 papers in recent years, with impressive numbers making progress on three or more questions. Slightly gentler questions in the fourth and fifth positions also gave the very best candidates time to think about the final problem, with a number going on to solve it successfully. At the other end of the distribution, it was good to see the vast majority of candidates getting into the 10^- regime on at least one question; only a handful of candidates did not seem to realise that full written solutions were required. A few responses to the geometry question consisted of nothing other than a diagram with every angle labelled as say $x, y, x+y, 90-x$ etc. This was particularly disheartening as those candidates almost certainly understood the relevant geometry, but since they neglected to explain the order in which the angles were found by *defining* one angle to be x and then *deducing* that some other angle must also be x and so on, they could not be given any credit. Other questions, notably 1, 5 and 6, naturally invited candidates to experiment with small examples and form conjectures. It was good to see most scripts engaging with this experimentation in a systematic way, and often spotting the correct patterns. However, this vital phase in solving such problems is of little value on its own. To score highly candidates needed to support their *observations* with *arguments* explaining why the observed patterns continued or the reasoning about small cases could be generalised.

The 2023 British Mathematical Olympiad Round 1 attracted 1675 entries. The scripts were marked in London (with some remote markers) from the 1st to the 3rd of December by a team of: Eszter Backhausz*, Sam Bealing*, Jonathan Beckett, Jamie Bell*, Robin Bhattacharyya, Damian Cheung, Laura Daniels, Stephen Darby, Chris Eagle, Ben Fairfax, Chris Garton, Anthony Goncharov, Aleksander Goodier, Amit Goyal, Aditya Gupta, Ben Handley*, Stuart Haring*, Jon Hart, Alexander Hurst, Ian Jackson*, Shavindra Jayasekera*, Vesna Kadelburg, Thomas Kavanagh, Jeremy King, Patricia King*, David Knipe, Hayden Lam, Larry Lau, Rhys Lewis, Warren Li, Sida Li, Thomas Lowe, Owen Mackenzie, Max Mackie, Sam Maltby, David Mestel*, Ana Meta Dolinar, Harry Metrebian*, Oliver Murray, Joseph Myers, Daniel Naylor, Martin Orr, Jenny Owladi, Preeyan Parmar*, Thomas Prince, Dominic Rowland, Heerpal Sahota, James Sarkies, Amit Shah, Gurjot Singh, Geoff Smith*, Rob Summerson, Stephen Tate, William Thomson, Velian Velikov, Tommy Walker Mackay, Zi Wang, Helen Xiaohui Chen, Tianyiwa Xie, Lingde Yang, Harvey Yau. (An asterix shows that the marker was a problem captain).

The problems were proposed by Geoff Smith, Geoff Smith, Sam Bealing, Dominic Yeo, Richard Freeland, and Ben Handley, respectively.

In addition to the written solutions in this report, video solutions can be found here.



The mean score was 20.9 and the median score was 21.

The thresholds for qualification for BMO2 were as follows:

Year 13: 43 marks or more.

Year 12: 42 marks or more.

Year 11: 39 marks or more.

Year 10 or below: 37 marks or more.

The thresholds for medals, Distinction and Merit were as follows:

Medal and book prize: 44 marks or more.

Distinction: 30 marks or more.

Merit: 13 marks or more.

Question 1

An unreliable typist can guarantee that when they try to type a word with different letters, every letter of the word will appear exactly once in what they type, and each letter will occur at most one letter late (though it may occur more than one letter early). Thus, when trying to type MATHS, the typist may type MATHS, MTAHS or TMASH, but not ATMSH.

Determine, with proof, the number of possible spellings of OLYMPIADS that might be typed.

Solution

The answer is $2^8 = 256$.

For a word with N letters, the condition of the problem is equivalent to the statement that for every positive integer $n < N$, the nth letter of the word is typed somewhere in the first $n + 1$ places.

In our problem, we take the letters of OLYMPIADS in order from the left and look at where they might appear in the typist's version. The letter O must appear either first or second: 2 possibilities.

Next the letter L must appear somewhere in the first three places. One of those three places has already been taken by the O, so there are 2 remaining possibilities for the position of the O, hence $2 \times 2 = 4$ possibilities for the positions of the O and L.

If we have identified the positions of the first $n - 1$ letters for some positive integer n satisfying $2 < n < 9$, we know that in the typist's version, the nth letter occurs somewhere in the first $n + 1$ places, and $n - 1$ of those have already been used for the first $n - 1$ letters, so there are 2 possibilities for the position of the nth letter.

This continues until we have identified the position of the first 8 letters. The final S must appear in the one remaining position, wherever that is.

So the total number of ways that the typist could type OLYMPIADS is $2^8 = 256$.

Remark

For every solution that considers possible locations at which a letter could be typed, there is another solution that considers possible letters that could be typed at a particular location.

For example, observe that there are two possible letters that can be typed last — D and S. Having chosen one, then of the three possible letters that can be typed second-last — A, D and S — there are two remaining possibilities. Continue backwards through the locations until we have selected a letter for each location.

Alternative

An alternative strategy is to relate the number of possibilities to the number of possible spellings of a shorter word. Let $f(n)$ be the number of valid spellings for a word of length n (without repeated letters); then the problem is to find $f(9)$. Given a word $ABCD\cdots$ of length n, consider all possible spellings of the shorter word $BCD\cdots$ of length $n - 1$. For each of the $f(n-1)$ spellings of the shorter word, we can obtain two spellings of the original word, by reinserting A either at the front or at the second place. These spellings are all different, because they differ either in the location of A or in the location of at least one letter from the shorter word. Furthermore, every spelling of the original word arises this way, because removing A from a valid spelling of the original word produces a valid spelling of the shorter word. Therefore $f(n) = 2f(n-1)$ for each $n \geq 1$. It is clear that $f(1) = 1$, and so $f(9) = 2 \cdot f(8) = 2^2 \cdot f(7) = \cdots = 2^8 \cdot f(1) = 2^8$. Alternatively, we could prove by induction that $f(n) = 2^{n-1}$, and then set $n = 9$.

Alternative

A variation considers which letter is typed first. If the first letter of an n-letter word is typed first then there are $f(n-1)$ ways to type the remaining letters. If the kth letter is typed first, then the first letter must be typed second, the second letter must be typed third, and so on, until we find that the $k-1$th letter must be typed at position k. We have now typed the first k letters in the first k locations. The final $n-k$ letters must therefore be typed in the final $n-k$ locations, and there are $f(n-k)$ ways to do this. We therefore have

$$f(n) = \sum_{k=1}^{n} f(n-k)$$

where $f(0) = 1$ since there is 1 way to type no letters. As before, this recursion could be used to directly find $f(9)$, or to prove by induction that $f(n) = 2^{n-1}$ for $n \geq 1$.

Alternative

A sophisticated strategy is to find a 1–1 correspondence (a bijection) between possible spellings and possible subsets of a suitable set of size 8. Since there are 2^8 subsets of a set of size 8, we can immediately deduce that there are 2^8 possible spellings. One such approach considers subsets of the first 8 letters, and places each of those letters one place late. We then prove that there is exactly one way to complete the spelling so that no other letter is typed late. Suppose that the letters that are *not* typed late are the letters at positions $j_1 < \cdots < j_k$ in the original word. Letters $1, \ldots, j_1 - 1$ are each typed one place late and therefore occupy places $2 \ldots, j_1$. Since letter j_1 is not typed late it can only occupy the first position. Next, letters $j_1 + 1, \ldots, j_2 - 1$ are each typed one place late, and so occupy positions $j_1 + 2, \ldots, j_2$. Since letter j_2 is not typed late, there is only one position remaining for it, namely position $j_1 + 1$. Arguing similarly for letters $j_3, \ldots, j_k$, we find that the only way to complete the spelling is to place letter j_ℓ at position $j_{\ell-1} + 1$ for $\ell = 2, \ldots, k$, and that this produces a valid spelling.

Markers' comments

The approach of iteratively choosing first where O is typed, then L and so on, was popular, as was the variation that chooses which letter is typed last, and then which letter is typed second-last, and so on. A complete solution must explain in which order the choices are being made, since the choices cannot be made independently. It must also justify why there are exactly 2 remaining choices at each stage. Many scripts lost marks for being unclear on one or both of these points.

Students must also reason that at the last stage there is only 1 remaining choice. Some failed to notice this, and concluded that the answer was $2^9 = 512$. This is a logical flaw and loses marks.

(Of course, students who correctly prove that the answer is 2^8, but mistakenly believe that this is equal to 512 are entirely forgiven, as are students who miscount the number of letters in the word $OLYMPIADS$ but otherwise have a correct argument.)

Many students found the number of possibilities for words of short lengths, correctly spotted a pattern, and predicted from this pattern what the answer would be for a word of length 9. Experimenting with small cases and formulating hypotheses is wise, and some marks were available for doing this; but an essentially complete logical argument was required to get a score beyond 3 marks.

Inductive approaches were also popular, and many were successful. Let there be $f(n)$ legal spellings of an n-letter word, then the common inductive steps were $f(n) = 2f(n-1)$ for $n > 1$ or $f(n) = f(n-1) + f(n-2) + \cdots f(1) + 1$ for $n > 1$. These can be obtained by working from inwards from either end of the word. The difficulty for markers was to make sure that such a recurrence was obtained by a proper argument, and not just by pattern spotting.

There were various types of subset approaches. For example, you can argue that the actual spelling of a word is determined by which of the first $n-1$ letters occur late. There are 2^{n-1} subsets of a set of size $n-1$, so that yields a solution.

A few students tried systematically to enumerate all possible spellings. This was usually not successful, either because some cases were missed, or because a pattern was claimed without justification.

Several students observed that there are 9! spellings in total if we ignore the typist's guarantee, and then attempted to subtract the number of spellings that violate the typist's guarantee. This is a legitimate line of attack, but most candidates trying this approach made calculation errors.

Question 2

The sequence of integers $a_0, a_1, \ldots$ has the property that for each $i \geq 2$, a_i is either $2a_{i-1} - a_{i-2}$ or $2a_{i-2} - a_{i-1}$.

Given that a_{2023} and a_{2024} are consecutive integers, prove that a_0 and a_1 are consecutive integers.

(Note that 6 and 7 are consecutive integers, as are 7 and 6.)

Solution

We would like to prove by induction that for all $0 \leq i \leq 2023$, a_i and a_{i+1} are consecutive.
Base case: a_{2023} and a_{2024} are consecutive.
Inductive step: Suppose that for some i, a_i and a_{i+1} are consecutive.
If $a_{i+1} = a_i + 1$ then either $a_{i+1} = a_i + 1 = 2a_i - a_{i-1}$, so $a_{i-1} = a_i - 1$ or $a_{i+1} = a_i + 1 = 2a_{i-1} - a_i$ so $a_{i-1} = (a_i + 1)/2$, contradiction, as all terms are integers. So $a_{i-1} = a_i - 1$.
Similarly, if $a_{i+1} = a_i - 1$ then either $a_{i+1} = a_i - 1 = 2a_i - a_{i-1}$, so $a_{i-1} = a_i + 1$ or $a_{i+1} = a_i - 1 = 2a_{i-1} - a_i$ so $a_{i-1} = (2a_i - 1)/2$, contradiction, as all terms are integers. So $a_{i-1} = a_i + 1$.
In either event, a_{i-1} and a_i are consecutive.
As a_{2024} and a_{2023} are consecutive and we showed that if a_i and a_{i+1} are consecutive then so are a_{i-1} and a_i, we have proved that a_0 and a_1 are consecutive.

Alternative

If a_0 and a_1 were equal, then all the terms of the sequence would be equal, which is not the case. So we can assume a_0 and a_1 are distinct. As all a_i are integers, $|a_1 - a_0| \geq 1$. The condition on the sequence can be expressed equivalently as the statement that for each $i \geq 2$:

$$a_i - a_{i-1} = a_{i-1} - a_{i-2} \text{ or } -2(a_{i-1} - a_{i-2})$$

In either case $|a_i - a_{i-1}| \geq |a_{i-1} - a_{i-2}|$. So

$$1 = |a_{2024} - a_{2023}| \geq |a_{2023} - a_{2022}| \geq \cdots \geq |a_1 - a_0| \geq 1$$

This implies $|a_1 - a_0| = 1$, which is to say that a_0 and a_1 are consecutive.

Markers' comments

Most successful candidates approached the problem by making a_{2022} the subject of the two relations. In doing so, they realised that only one of the relations results in an integer value and also that this value and a_{2023} must be consecutive. At this point, a carefully phrased induction (applying the same ideas to a_{2023} and a_{2022} to find a_{2021}, and so on) solved the problem. Some candidates lost out on marks for not stating why only one of the relations can be used in every step, not just the first one. Candidates were not penalised for phrasing their induction informally but we recommend they look up the correct terminology.

Many candidates calculated options for a_{2021} instead. This method needed a bit more algebra, but worked similarly, except that it only proved that terms of the form a_{2i-1} and a_{2i} are consecutive. Some didn't spot that a_0 and a_1 are not of this form, and so lost out on marks for not discussing the last step needed from a_1 and a_2 to a_0 and a_1.

Candidates considering the difference between consecutive terms were usually successful but often lost marks as they presumed that $|a_i - a_{i+1}| \leq 1$ implied the terms were consecutive without clearly excluding 0 and non-integer values.

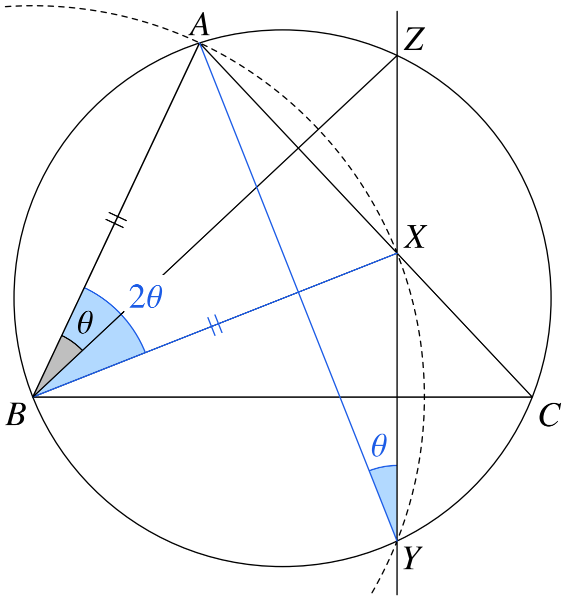
Some candidates misunderstood the question and presumed that the same relation must be used for the whole sequence, as opposed to a potential combination of both, others only considered one. A few assumed that a_{2023} and a_{2024} are consecutive *increasing*, despite the note. These mistakes stopped candidates from scoring highly.

Question 3

Let ABC be a triangle with $\angle ACB < \angle BAC < 90°$. Let X and Y be points on AC and the circle ABC respectively such that $X, Y \neq A$ and $BX = BY = BA$. Line XY intersects the circle ABC again at Z.

Prove that BZ is perpendicular to AC.

SOLUTION



To construct an accurate figure for this question we must note that A, X and Y all lie on a circle This means, amongst other things, that triangle ABX is isosceles with apex B.

We let $\angle ZBA = \theta$ as shown.

We add segments BX and AY (shown in blue).

Now $\angle ZYA = \theta$ (Angles in the same segment in circle $ABYZ$.)

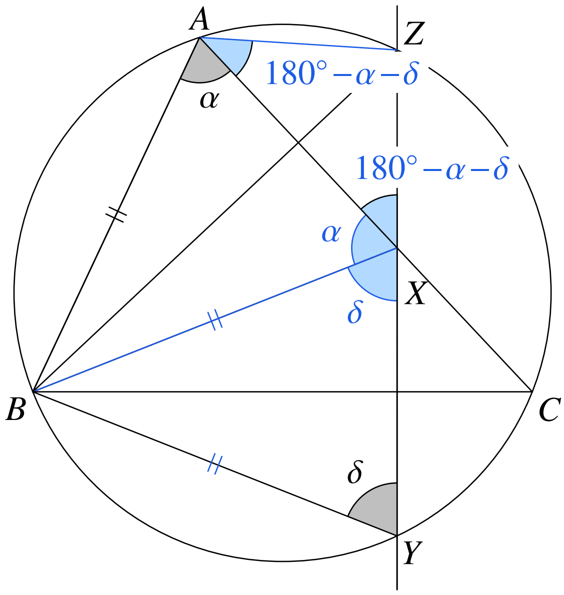
Also $\angle XBA = 2\theta$ (Angle at centre 2× angle at circumference in circle AYX.)

Thus BZ is an (internal) angle bisector in the (isosceles) triangle XBA, so BZ is perpendicular to XA.

REMARK

The fact that in an isosceles triangle the angle bisector from the apex is an altitude can be quoted at BMO level. It is easy to prove by checking that it splits the triangle into two triangles that are congruent by SAS.

Alternative



We let $\angle BAC = \alpha$ and $\angle XYB = \delta$ as shown.

We add segments BY, BX and AZ (shown in blue).

Now $\angle BXY = \delta$ (Isos triangle XBY.)

$\angle AXB = \alpha$ (Isos triangle ABX.)

$\angle ZXA = 180° - \alpha - \delta$ (Straight line YXZ.)

$\angle BAZ = 180° - \delta$ (Cyclic quad $ABYZ$.)

So $\angle XAZ = 180° - \alpha - \delta$ (Angles at A.)

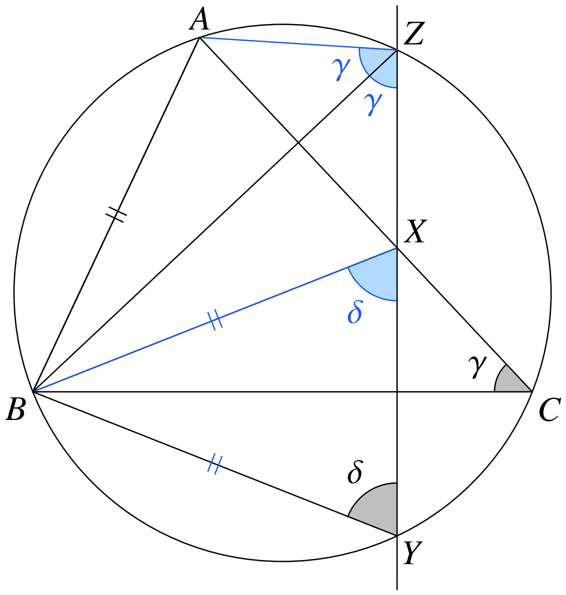
This means triangle XZA is isosceles, so $ZA = ZX$.

This, combined with the fact that $BA = BX$ means $ABXZ$ is a kite, so its diagonals, BZ and AC are perpendicular.

Remark

The fact that diagonals of a kite a perpendicular can be quoted. It can be proved by first showing the two non-isosceles halves of the kite are congruent using SSS, then using the fact that the angle bisector of an isosceles triangle is an altitude as in the first solution.

Alternative



We let $\angle ACB = \gamma$ and $\angle XYB = \delta$ as shown.

We add segments BY, BX and AZ (shown in blue).

Now $\angle BXY = \delta$ (Isos triangle XBY.)

$\angle AZB = \gamma$ (Angles is same segment in circle $ABCZ$.)

$\angle BZY = \gamma$ ($AB = BY$ and equal angles subtend equal arcs.)

$\angle BAZ = 180° - \delta$ (Cyclic quad $ABYZ$.)

So $\angle ZXB = 180° - \delta$ (Straight line ZXY.)

Now triangles ABZ and XBZ have two, and therefore three, angles in common. The have a common side BZ so are congruent by ASA. This means $ABXZ$ is a kite and we conclude as before.

Remark

Having found the equal angles at Z, the common side BZ and the equal sides BA and BX it is tempting to say that ABZ and XBZ are congruent by ASS. However, this is not a congruence condition. Indeed triangle BYZ has the same common angle and sides.

Markers' comments

Those who were most successful at this problem started by drawing a large, accurate diagram with a compass and ruler as shown in the first solution. This helped them to make conjectures around other equal angles and lengths which many were able to turn into a complete solution.

The solutions given above provide a useful template for writing up a BMO1 geometry problem. Having two columns, one for angle equalities and one for the justification, ensures that each result is clearly justified. Many lost substantial marks by failing to provide sufficient justification for all of their steps, for example using the fact that equal chords subtend equal angles without justification.

Question 4

Find all positive integers n such that $n \times 2^n + 1$ is a square.

Solution

Let $a > 0$ be an integer such that $n \times 2^n + 1 = a^2$. It follows that $2^n n = (a-1)(a+1)$. The integers $a-1$ and $a+1$ have the same parity and their product is even, so both must be even. Moreover, since $a-1$ and $a+1$ are consecutive even integers, one of them is twice an odd number, so the other divisible by 2^{n-1}.

Thus $a+1 \geq 2^{n-1}$ which means $a-1 \leq 2n$. This implies that $2n+2 \geq a+1 \geq 2^{n-1}$.

We claim that this false for $n \geq 5$, so we only need to check $n = 1, 2, 3$ and 4.

When $n = 5$, we have $12 = 2n+2 < 2^{n-1} = 16$, so the claim holds for $n = 5$.

If we know that $2k+2 < 2^{k-1}$ for some positive k, we can consider:

$$2(k+1)+2 = (2k+2) \times \frac{k+2}{k+1} < 2^{k-1} \times \frac{k+2}{k+1} < 2^{k-1} \times 2 = 2^{(k+1)-1}$$

so if the claim holds for k, it holds for $k+1$.

Checking $n = 1, 2, 3$ and 4, we find that the values of $n \times 2^n + 1$ are $3, 9, 25$ and 65 respectively. Hence $n = 2$ and $n = 3$ work and $n = 1, 4$ do not.

Markers' comments

This was quite an approachable question 4, and there were many good attempts at it. Many rearranged and factorised using a difference of squares. Some deduced that the square must be odd, so used a different designation for their unknown and simplified an alternative expression before factorising.

Either approach led to an equation of the form $n2^{n-2} = m(m+1)$ or something equivalent to this. Many incorrectly deduced that n and 2^{n-2} must be m or $m+1$ in some order. This received little further credit as it misses the possibility that the factors of n are split between m and $m+1$. It was important to note the crucial point about m and $m+1$ being coprime. However, if $n = pq$ it is possible that $p \mid m$ and $q \mid m+1$. What can be deduced is that m and $m+1$ cannot both contain factors of 2, so either $2^{n-2} \mid m$ or $2^{n-2} \mid m+1$.

From here, both cases need to be considered to complete the proof, and those who only considered one were penalised. Some considered both and then used one having demonstrated it as a worst-case to continue the proof, which was acceptable.

To finish the proof, it was necessary to bound a suitable expression to show equality was impossible above that bound. Many noted that "exponentials beat linear expressions", but that is only true beyond a certain point, which needed to be noted and demonstrated. A more formal inductive proof was the best approach. Those using graphs or considering rates of change needed sufficient detail to get full credit.

Lastly, all those values below the bound needed to be checked manually. There was little penalty for failing to explicitly show this had been done and many just wrote the values which worked, but it is good practice to show all of them to complete the solution.

Question 5

An artist arranges 1000 dots evenly around a circle, with each dot being either red or blue. A critic looks at the artwork and counts *faults*: each time two red dots are adjacent is one fault, and each time two blue dots are exactly two apart (that is, they have exactly one dot in between them) is another.

What is the smallest number of faults the critic could find?

Solution

Placing red then blue dots two at a time round the circle ($\ldots RRBBRRBB \ldots$), gives 250 'red' faults and no 'blue' faults.

It remains to prove that every possible configuration contains at least 250 faults.

In every five consecutive dots there must be at least one fault (if any of the middle 3 dots are red, then either that dot has a red neighbour or both its neighbours are blue, both giving a fault; if none of the middle 3 dots are red, then we have 2 blue dots that are 2 apart, giving a fault).

Now divide the 1000 dots into blocks of five consecutive dots, each block overlapping by one dot at each end. There are 250 such blocks, each containing at least one fault. Since each fault consist of at least two dots, and blocks overlap by at most one dot, it is impossible to count the same fault in two blocks. Hence, there must be at least 250 faults.

A slight variation on this argument is to divide the 1000 dots into non-overlapping blocks of four dots and argue that each such block either contains a fault, which we will call an *internal* fault, or is $RBBR$. In the latter case then the dot that comes directly after the block will be the second dot in a red fault if it is red and will be the second dot in a blue fault if it is blue. Either way we have a fault the first of whose dots is in our block, which we call an *external fault*. Since each of the 250 blocks contributes either an internal or an external fault, there must be at least 250 faults in any configuration.

Alternative

There are various alternative approaches to showing every configuration has at least 250 faults, such as the following: without loss of generality, we may consider a configuration with the minimum number of faults and no more than two consecutive dots the same colour. Indeed, if there are three or more consecutive blue dots, replace one in the middle with a red dot (this doesn't add any faults, and might remove some faults made of two blue dots that are two apart); if there are three or more consecutive red dots, replace one in the middle with a blue dot (this removes two faults made of two adjacent red dots, adds at most two faults made of blue dots two apart). This step increases the number of blocks of consecutive dots of the same colour, so it must terminate after finitely many steps, at which point all such blocks have length at most 2.

Now that all blocks of consecutive dots the same colour have length 1 or 2, the number of faults equals the number of such blocks of red dots (a block of two red dots has a fault in it; a block of one red dot gives a fault of the two blue dots either side). Since there are at least 500 blocks, of alternating colours, there are at least 250 blocks of red dots, so at least 250 faults.

Markers' comments

This question consisted of two parts: firstly finding a configuration of dots that has 250 faults and secondly showing that any configuration of dots admits at least 250 faults. The first part was very approachable for a question 5. Indeed, of the candidates that attempted this question, approximately 80% managed to find the optimal configuration of $RRBBRRBB\ldots$ repeated around the circle. Many candidates that did not stumble across this solution offered configurations which repeated every three dots such as $BRBBRB\ldots$ or $RBBRBB\ldots$ with 333 or 334 faults; these received partial credit.

However, the majority of candidates struggled to prove the lower bound for the number of faults is 250. Many assumed without justification that an optimal configuration must consist of a repeating pattern and therefore any attempts at showing a lower bound did not apply to all possible configurations of dots. Some candidates tried an approach via induction on the number of dots but most of these attempts did not properly justify how the lower bound on the number faults increases in the inductive step.

Several candidates made the promising observation that $RBBR$ is the only sequence of four dots that do not contain a fault and some went further to show that the sequence $RBBR$ is always followed by a fault. However, most candidates made an incorrect logical leap to assume that since $RBBR$ is the longest sequence without a fault, an optimal solution must repeat $RBBR$. Whilst it is true in this case that the optimal solution is $RBBR$ repeated around the circle, to show a lower bound on the number of faults, one must show that *any* configuration must incur at least 250 faults. Note that candidates that stated claims regarding $RBBR$ without full justification were penalised.

In order to receive full marks for this approach, following an observation of the optimality of $RBBR$, a candidate needed to divide the 1000 dots into 250 blocks of fours (non-overlapping) or fives (overlapping) and conclude that there must be a fault that either begins in the block (in the case of non-overlapping fours) or contained within the block (in the case of overlapping fives); alternatively it was possible to base an argument on showing that counting around any arrangement we must see a fault at least every four dots but for full credit this needed to be explained clearly and carefully, which was rare.

Finally, for those who are interested, consider what the optimal solution would be for a general n dot circle. Alternatively, consider the problem where we define a fault differently: for instance, what if we have a fault when two blue dots are exactly three apart (instead of two)?

Question 6

For some integer $n > 4$ a convex polygon has vertices $v_1, v_2, \ldots, v_n$ in that cyclic order. All its edges are the same length. It also has the property that the lengths of the diagonals $v_1v_4, v_2v_5, \ldots, v_{n-3}v_n, v_{n-2}v_1, v_{n-1}v_2$ and v_nv_3 are all equal.
For which n is it necessarily the case that the polygon has equal angles?

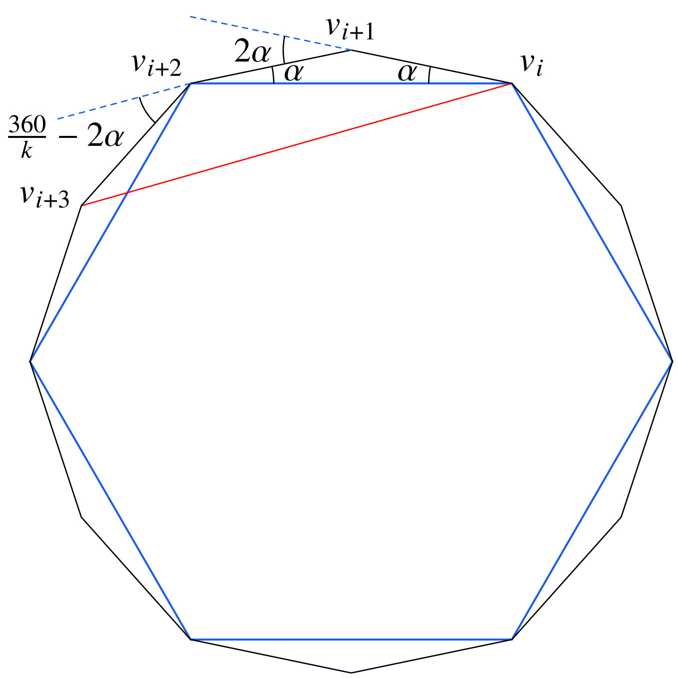
Solution

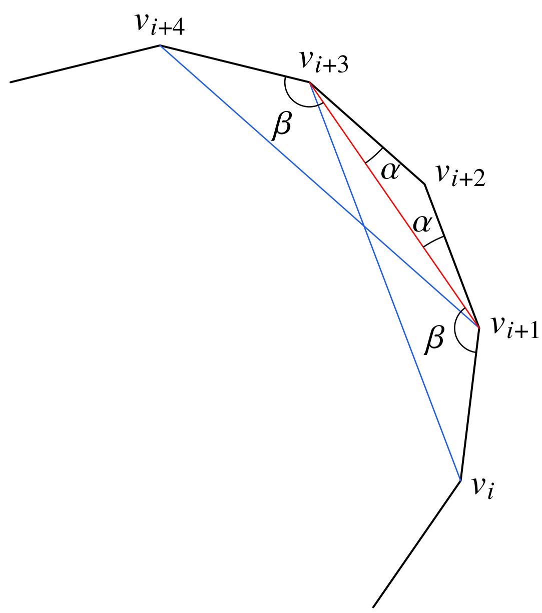
It is necessary and sufficient that n is odd.

If $n = 2k$ we can construct a counterexample as follows: begin with a regular k-gon (shown in blue in the left hand figure, where $k = 6$) and erect an isosceles triangle with angles $\alpha, \alpha, 180 - 2\alpha$ externally on each side. This creates an equilateral $2k$-gon whose external angles are alternately, 2α and $360/k - 2\alpha$ (shown in black in the figure).

If we choose α small enough that $2\alpha < 360/k - 2\alpha$ this polygon will not be regular. It will also be convex since its interior angles are $180 - 2\alpha < 180$ and $180 - (360/k - 2\alpha) < 180 - 2\alpha < 180$.

To check it satisfies the condition on the diagonals we note that each v_i-v_{i+3} diagonal (shown in red in the figure) is the side of triangle with one side joining consecutive vertices on the $2k$-gon, and another side joining two vertices of the k-gon we started with. The included angle is equal to $180 - 360/k + \alpha$, so all such triangles are congruent by SAS.





Now suppose n is odd. Consider the vertices $v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}$ (with subscripts mod n).

In the right hand figure the triangles $v_iv_{i+1}v_{i+3}$ and $v_{i+4}v_{i+3}v_{i+1}$ each have a black, a red and a blue side. They are therefore congruent by SSS. We call the the (equal) angles opposite the blue sides β. That is, we let $\angle v_{i+3}v_{i+1}v_i = \angle v_{i+4}v_{i+3}v_{i+1} = \beta$.

Since $v_{i+1}v_{i+2}v_{i+3}$ is isosceles (with two black sides and one red) we may call its (equal) base angles α. That is, we let $\angle v_{i+2}v_{i+1}v_{i+3} = \angle v_{i+1}v_{i+3}v_{i+2} = \alpha$. Now it is clear that the angles in polygon at v_{i+1} and v_{i+3} are both equal to $\alpha + \beta$. This holds for every i.

Thus $\angle v_n v_1 v_2 = \angle v_2 v_3 v_4 = \cdots = \angle v_{n-1} v_n v_1 = \angle v_1 v_2 v_3 = \cdots = \angle v_{n-2} v_{n-1} v_n$.

Therefore, all the angles of the polygon are equal.

Markers' comments

As usual, Q6 proved to be a very challenging question, despite the solution being potentially quite short. This might be because, as a 'non-standard geometry' problem, many students may have found the required reasoning to be unfamiliar.

Many students managed to solve the $n = 5$ case, but points were only given for statements that apply more generally. Many other students were unsure about the exact meaning of 'convex'. The word was included in the problem statement in order to help students by eliminating any strange diagrams that would otherwise need to be considered, but it did cause considerable confusion.

Another common problem was students fixating on the value of n mod 3, noticing that when $n \equiv 0 \pmod 3$ the given diagonals form into three disjoint $n/3$-gons. While true, this observation didn't actually lead to a solution and such scripts generally scored 0 unless there were other observations that could lead to a solution.

Many other students were held back from full marks by the weakness in their handling of the even case. Implicitly, a question such as this requires a proof that for all odd n the polygon must be regular, and that for all even n it need not be regular. Concluding 'therefore it must be regular for all odd n' is only half of the solution and will not score close to full marks. To score a 10, the even case has to show that the construction is a) a polygon, b) convex, and c) satisfies the length constraints. As a result, very few scripts scored full marks.

Finally, for those who are interested in this style of question, you may like to try solving it with a different set of constraints. The problem as set could be called $\{1, 3\}$ as 'diagonals' of step size 1 and 3 are constrained. Can you solve $\{2, 3\}$ or $\{1, 4\}$? To our knowledge, they are open questions.